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Research Article

Portfolio Theory for α-Symmetric and Pseudoisotropic Distributions: k-Fund Separation and the CAPM

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The shifted pseudoisotropic multivariate distributions are shown to satisfy Ross' stochastic dominance criterion for two-fund monetary separation in the case with risk-free investment opportunity and furthermore to admit the Capital Asset Pricing Model under an embedding in L^{α} condition if $1 < \alpha \le 2$, with the betas given in an explicit form. For the α-symmetric subclass, the market without risk-free investment opportunity admits $2d$-fund separation if $\alpha = 1 + 1/(2d - 1)$, $d \in \mathbb{N}$, generalizing the classical elliptical case $d = 1$, and we also give the precise number of funds needed, from which it follows that we cannot, except degenerate cases, have a CAPM without risk-free opportunity. For the symmetric stable subclass, the index of stability is only of secondary interest, and several common restrictions in terms of that index can be weakened by replacing it by the (no smaller) indices of symmetry/of embedding. Finally, dynamic models with intermediate consumption inherit the separation properties of the static models.

1. Introduction

Portfolio separation, that is, the property of reducing the dimension of a portfolio optimization problem to a low number of vectors ("funds") without welfare loss to the agents in question, has been treated extensively since Tobin [1]. There are two main directions: the one which is the subject of this paper is the characterization of those returns probability distributions for which those funds will do for all agents. The other is the characterization of preferences which admit the property for all suitable returns distributions (the standard work being Cass and Stiglitz [2], but see even the modern probabilistic approach of Schachermayer et al. [3]); there are also other routes to the separation property, for example, risk measures, falling somewhat in between beliefs and preferences (contributions include this author [4] and independently Giorgi et al. [5]).

This paper concerns the distributional side of the theory, where the standard literature reference is Ross [6]. Ross considers preferences compatible with second-order stochastic dominance (and in footnotes, preferences merely assumed compatible with first-order dominance). The core of his result is the property that the *returns distribution vector* be such that the *portfolio* returns distributions (univariate) can be ordered by their mean once a single dispersion parameter is given, and for the second-order case: by their dispersion once the mean is given. Subsequently, Owen and Rabinovitch [7] and Chamberlain [8] establish that the elliptical (also frequently referred to as "elliptically contoured") distributions satisfy Ross' conditions for two-fund separation. Their setting is a mean-variance tradeoff, tying the knot back to the Markowitz [9] approach as employed by Tobin [1]. Over these decades, the development has offered surprises to quite a few of the giants who bear today's theory on their shoulders: Markowitz turned out predated by more than a decade by de Finetti [10] (see Markowitz' account [11] where he also credits Roy [12]). Tobin conjectured that any two-parameter portfolio returns distribution family would admit two-fund separation; counterexamples were given by Samuelson [13], Borch [14], and Feldstein [15]. Fama's discovery ([16], can also be read out of Samuelson [17]) that vectors of iid symmetric α-stables admitted two-fund separation led Cass and Stiglitz to suggest that α-stability was even *necessary*, making a reservation for independence, which quickly turned out to be a most appropriate one, as Agnew [18] provided a nonstable example. However, the properties that enabled Owen, Rabinovitch, and

Chamberlain to verify the Ross [6] criterion for the ellipticals were to be found as far back as Schoenberg [19, 20] in 1938, before modern portfolio theory.

The classical (elliptical) 2-fund separation result holds irrespective of whether a "risk-free" numéraire opportunity exists, and one of the funds can be chosen to be the safest available (the "minimum variance" portfolio, so the risk-free case admits so-called "monetary separation"). This paper considers the generalization to the (shifted) so-called *pseudoisotropic distributions*, a multivariate class of symmetric random variables such that all linear combinations of the coordinates are of the same type. The pseudoisotropic distributions admit a dispersion quasinorm ς (often called the "standard") which is symmetric and positively homogeneous and which, together with the excess returns entering via a location shift, characterizes the portfolio return distribution completely, which is briefly summarized as follows:

(i) *With* risk-free opportunity, the (shifted) pseudoisotropic distributions admit two-fund monetary separation just like the elliptical subclass, or like the subclass of iid symmetric α-stable random variables as established already by Fama [16] (for results in continuous time: this author [21] and Ortobelli et al. [22]).

Furthermore, we have a CAPM if ς is differentiable outside the origin; a fortiori, so is the case for the so-called α-symmetric distributions with $\alpha > 1$, and for the symmetric stables with index of stability > 1, but also for certain nonintegrables.

(ii) *Without* risk-free opportunity, separation will only be admitted by a few special cases leading to $2d$-fund separation if the index of symmetry is one of the values $\alpha = 1 + 1/(2d - 1)$, $d \in \mathbb{N}$, that is, one-and-an-oddth, where $d = 1$ subsumes the elliptical distributions $\alpha = 2 = 1 + 1/1$.

Also the CAPM breaks down except in elliptical or degenerate cases. Fama [23, section VI.B] remarks that the presence of risk-free opportunity "greatly simplifies determination of the efficient set of portfolios," and indeed, for the $2d$-fund separation cases just mentioned, the efficient set is no longer convex.

The paper will first introduce terminology and then in Section 3 state the single-period market model and review stochastic dominance. Section 4 will introduce pseudoisotropic random variables, and Sections 5–7 will point out how/when they admit, respectively, monetary separation, separation without risk-free opportunity, and CAPM. Section 8 sketches how a dynamic model inherits the separation properties from the static model.

2. Notation, Terminology, and Standing Assumptions

We work in $\mathbb{R}^D$ for arbitrary finite $D \geq 2$; some results will be vacuous for low D. Random quantities are denoted by Latin letters (boldfaced if vector-valued). Minuscles (Greek/Latin, vectors if bold) are either nonrandom or choice variables; a D-vector $\boldsymbol{\xi}$ is called a *portfolio* if it takes values in a given set to be denoted by H or L (notation to depend on shape; "unrestricted" if no such set restriction is given). $\mathbf{1}$ is the vector of ones, and $\mathbf{0}$ is the null vector. Vectors are columns by default, unless indicated by superscript "$^\top$" (transposition) or given as a gradient. We apply the *signed power* notation $x^{\langle p \rangle} := |x|^p \operatorname{sign}(x)$ even to vectors, elementwise: $\mathbf{x}^{\langle p \rangle} = (x_1^{\langle p \rangle}, \dots, x_D^{\langle p \rangle})^\top$. (Notice that $\mathbf{x} \mapsto \mathbf{x}^{\langle p \rangle}$ is invertible.) Matrices are Greek uppercase boldfaced letters.

A set $H \supsetneq \{\mathbf{0}\}$ is *radial* if it is composed as a union of half-lines from the origin: $\mathbf{x} \in H \Leftrightarrow q\mathbf{x} \in H \ \forall q > 0$. Constraining the portfolio to the closed first orthant models a "no short sale" constraint, and we will use that terminology as well. No short sale on some, but not all, investment opportunities will also correspond to a radial constraint. As commonplace in the literature, we will frequently refer to the numéraire as the "risk-free" investment opportunity and the other investments as "risky."

The $\sim$ symbol denotes equal probability law. A random variable and its distribution are *symmetric* if $\mathbf{X} \sim -\mathbf{X}$; then $\boldsymbol{\mu} + \mathbf{X}$ is called *shifted symmetric* for nonrandom $\boldsymbol{\mu}$.

Assumption 1. We will allow for constraints to be specified (in the single-period model, we will consider either the constraint to a radial set, covering, e.g., no short sale conditions, to an affine half-space representing no borrowing or limited degree of leverage, or to the affine hyperplane of no risk-free opportunity). After having restricted the opportunity set according to these constraints, we will assume the market to be *free from arbitrage opportunities and from redundant investment opportunities*. (If there is a redundant opportunity, then we can leave it out and rebuild the model without it.) In particular, the independent radial scalings R_0 and R are never Dirac at zero; if any of these is constant, it is without loss of generality $= 1$.

Note that, in line with the literature on portfolio separation, we do not assume limited liability, which in fact holds only in a few well-known cases, all elliptical.

3. The Single-Period Market and the Preferences

Consider a single-period investment allocating wealth w between $\boldsymbol{\xi}$ in $D \in \mathbb{N}$ "risky" investment opportunities and the remaining $w - \mathbf{1}^\top\boldsymbol{\xi}$ in a numéraire (enumerated as the 0th coordinate) that returns X_0 per monetary unit invested. Writing the risky returns vector as $X_0\mathbf{1} + \boldsymbol{\mu}R_0 + \mathbf{X}R$ for some nonrandom *location parameter* $\boldsymbol{\mu}$ (resembling a representation common for elliptical distributions, e.g., Cambanis et al. [24]), the *portfolio return* then becomes our model *ansatz*

$$wX_0 + \boldsymbol{\xi}^\top \left(\boldsymbol{\mu}R_0 + \mathbf{X}R\right). \tag{1}$$

$(R_0, R, \mathbf{X})$ will be specified conditional on X_0, with (R_0, R) conditionally independent of $\mathbf{X}$. We will later assume $\mathbf{X}$ to be symmetric (but not that it is integrable!). It will represent no loss of generality to interpret, or even formally assume,

X_0 as "risk-free"; we say that *a risk-free opportunity exists* unless all agents are constrained to $\mathbf{1}^\top \boldsymbol{\xi} = w$.

We first define *agents* and then separation (to hold over all agents); note that "k-fund separation" implies $k+1$-fund separation; Theorem 11 will specify when a result cannot be improved upon.

Definition 2. To compare X^* and X, suppose $\widehat{X} \geq 0$ a.s. and consider the formula

$$X^* + \check{X} \sim X + \widehat{X}. \tag{2}$$

(i) By an *agent* we mean a pair of initial *wealth* $w \in \mathbb{R}$ and a partial *preference* ordering over random variables, where preferences are such that X^* is always (weakly) preferred to X whenever (2) holds with $\check{X} = 0$ a.s. for some $\widehat{X} \geq 0$ a.s., in which case we say that X^* (weakly) *first-order stochastically dominates* X.

(ii) An agent is *risk-averse* if X^* is always (weakly) preferred to X even whenever we admit any $\check{X}$ which is independent of the three others and symmetric.

(iii) Suppose that there exist $k \geq 1$ vectors ("funds") $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_k$ such that, for any given portfolio $\boldsymbol{\xi}$, there exist $q_1, \ldots, q_k$ so that return (1) is weakly first-order stochastically dominated by the return obtained using in place of $\boldsymbol{\xi}$ in (1) the portfolio

$$\boldsymbol{\xi}^* = q_1 \boldsymbol{\varphi}_1 + \cdots + q_k \boldsymbol{\varphi}_k. \tag{3}$$

Then we say that the returns distribution admits $k+1$-*fund monetary separation* if a risk-free opportunity exists (fund $k+1$ being the numéraire) and k-*fund separation* if numéraire holdings $w - \mathbf{1}^\top \boldsymbol{\xi}^*$ vanish identically for all agents.

Remark 3. Notice first that we do not assume that each agent has an "optimal" (finite) portfolio; rather, the property says that for any given portfolio there is one which is at least as good and which uses only the funds (implying that the restriction to the funds is without welfare loss).

Item (i) is the so-called *mass-transfer* criterion for first-order stochastic dominance. It is equivalent to either of the following; see, for example, Østerdal [25] for more on the various definitions: $\mathrm{CDF}_{X^*} \leq \mathrm{CDF}_X$, or $\mathsf{E}[u(X^*)] \geq \mathsf{E}[u(X)]$ for every bounded nondecreasing (i.e., "utility") function u. We will frequently use that if neither X^* first-order dominates X nor vice versa, then there are two agents which disagree over preference between them; indeed, the utility function $1_{x \geq \widehat{x}}$ prefers X^* to X iff $\mathrm{CDF}_{X^*}(\widehat{x}) \leq \mathrm{CDF}_X(\widehat{x})$. Notice that there is no first-order dominance between X and $\mu + \sigma X$ if X is real and symmetric and has full support and $\sigma > 1$ and $\mu \geq 0$; if one can increase σ without decreasing μ, then there is some agent who will prefer it and some who will not.

Second-order stochastic dominance corresponds to risk aversion, but in contrast to the common literature, which assumes sufficient integrability for $\mathsf{E}\check{X} = 0$ and u concave, we merely ask if an agent will reject any independent symmetric noise. Risk aversion is not a main point of this paper and will be invoked only in a few instances, where they can do with fewer funds: As is well-known, all risk-averse agents can do with the fund $\mathbf{1}$ if there is no risk-free opportunity and, for example, all returns are iid Gaussian, but a non-risk-averse agent could need another fund to boost variance. Theorem 11 will touch this issue.

Taking the well-known Gaussian as example, monetary two-fund separation is due to the following features, assuming for simplicity $X_0 = 0$: the set of all possible portfolio returns is a family wherein each distribution is fully characterized by location (which is a good to every agent!) and scale. Both these functionals are homogeneous of degree one, so if $\boldsymbol{\varphi}$ maximizes location given scale (standard deviation) of 1, then $Q\boldsymbol{\varphi}$ maximizes location given a scale of Q, and for every $\boldsymbol{\xi}$ the return is first-order stochastically dominated by $Q\boldsymbol{\varphi}$ for the appropriate Q. The next section will introduce the more general class of pseudoisotropic distributions which, when shifted by a location, share these features. What will not carry over, except in a much weaker result valid in exceptional cases, is separation under restriction to an affine subspace not containing $\mathbf{0}$.

4. Pseudoisotropic and α-Symmetric Distributions

The pseudoisotropic random vectors form a multivariate distribution class which contains, among others, the symmetric ellipticals (and no other square-integrable distributions!) and symmetric α-stables. The following will give a primer on the theory assuming the basics of these subclasses are known (see, e.g., Cambanis et al. [24] and the beginning of Samorodnitsky and Taqqu [26]). For the idea of finding D-dimensional versions of univariate distributions, see Eaton [27]; the term pseudoisotropic does refer to the multivariate $\mathbf{X}$; see Jasiulis and Misiewicz [28, Definition 3].

Definition 4. A symmetric distribution in $\mathbb{R}^D$ is called *pseudoisotropic* if, for some order 1 positive-homogeneous *standard* $\varsigma : \mathbb{R}^D \to [0, \infty)$ and some (complex) function h, the characteristic function can be represented as $q\boldsymbol{\theta} \mapsto \mathsf{E}[e^{iq\boldsymbol{\theta}^\top \mathbf{X}}] = h(|q|\varsigma(\boldsymbol{\theta}))$.

Thus we have $\boldsymbol{\theta}^\top \mathbf{X} \sim \varsigma(\boldsymbol{\theta})\widetilde{X}$ for some marginal $\widetilde{X}$ (any non-Dirac marginal X_i will do!), a property obviously preserved under matrix transformations. Pseudoisotropy generalizes ellipticity located at zero (then, ς^2 is a quadratic form), but ellipticity admits some special properties. For example, nonelliptical pseudoisotropic distributions cannot have finite second-order moments and must be absolutely continuous with respect to D-dimensional Lebesgue measure except a possible point mass at the origin, or a marginal being Dirac. Those exceptions can be done away with the latter by the assumption of no arbitrage or no second risk-free opportunity and the former by incorporating it in R. There is thus no loss of generality in the following restriction to what Misiewicz [29, Remark II.2.1] calls "pure" pseudoisotropic measures.

Assumption 5. For the pseudoisotropic distributions considered in this paper, no coordinate has any point mass at the origin, and $\varsigma(\boldsymbol{\theta}) = 0$ only iff $\boldsymbol{\theta} = \mathbf{0}$.

Pseudoisotropy also generalizes symmetric α-stability; indeed, if any two coordinates of a pseudoisotropic variable are independent, we do have symmetric stability. Like for symmetric stable distributions, there are some geometric properties to observe. We introduce some terminology, compare, for example, Koldobsky [30, 31] (wherein the reader also can find why the restriction to $p \leq 2$ does not rule out any interesting cases for our purposes) and Kalton et al. [32].

Definition 6. An *origin-symmetric star body* K in $\mathbb{R}^D$ is an origin-symmetric compact with a continuous boundary crossed precisely twice by each line through $\mathbf{0}$, required interior to K. Let the K-quasinorm $\|\cdot\|_K$ be the (well-defined!) associated Minkowski functional $\|\boldsymbol{\theta}\|_K = \min\{a > 0;\ \boldsymbol{\theta}/a \in K\}$. Fix $p \in [0, 2]$; we say that the quasinormed space $(\mathbb{R}^D, \|\cdot\|_K)$ *embeds* in L^p if $\|\cdot\|_K$ admits a so-called Blaschke-Lévy representation

$$\|\boldsymbol{\theta}\|_K = \begin{cases} \left(\displaystyle\int \left|\boldsymbol{\theta}^\top \mathbf{x}\right|^p \varkappa(d\mathbf{x})\right)^{1/p}, & p \in (0, 2], \\ \exp \displaystyle\int \ln \left|A\boldsymbol{\theta}^\top \mathbf{x}\right| \varkappa(d\mathbf{x}), & p = 0 \end{cases} \tag{4}$$

for some finite *spectral measure* $\varkappa$ supported by the unit sphere (necessarily symmetric, and for $p = 0$ it integrates to one) and for $p = 0$ some $A > 0$ (explicitly computed in terms of $\varkappa$ in [32, p. 3-4]). The supremum over those $p \in [0, 2]$ such that (4) holds seems not to have an established term: we call it the *embedding index*.

Notice that if $p > 1$, then $\|\boldsymbol{\theta}\|_K$ is strictly quasiconvex, with (by bounded convergence) gradient $= \int (\boldsymbol{\theta}^\top \mathbf{x})^{\langle p-1\rangle} \mathbf{x}^\top d\varkappa$ continuous for $\boldsymbol{\theta} \neq \mathbf{0}$.

The connection to pseudoisotropic variables is the following fact, conjectured by Lisitskiĭ [33] and later proven by Koldobsky [31, Corollary 1], that for our purposes we do have embedding in L^0 (and thus the embedding index is a well-defined number $\in [0, 2]$).

Theorem 7. *For any pseudoisotropic* $\mathbf{X}$ *in* $\mathbb{R}^D$, ς *is the Minkowski functional* $\|\cdot\|_K$ *of some origin-symmetric star body* K*, such that* $(\mathbb{R}^D, \|\cdot\|_K)$ *embeds in* L^0.

Embedding in L^p for $p \in (0, 2]$ implies embedding in L^α for all $\alpha \in [0, p]$, each with its own spectral measure, henceforth the *α-spectral measure* if needed to distinguish; a stronger assertion than Theorem 7 is therefore the Misiewicz conjecture that the embedding index is > 0. This question remains open, though potential counterexamples must satisfy restrictive conditions (see [30]). In particular, they must be extremely tail-heavy, as it is well-known that nonembeddability in L^p ($p \in (0, 2]$) implies infinite pth-order moment.

The elliptical distributions located at the origin are precisely the ones which are pseudoisotropic and embed in L^2. The Blaschke-Lévy representation (4) then takes the form $(\boldsymbol{\theta}^\top \int \mathbf{x}\mathbf{x}^\top d\varkappa\, \boldsymbol{\theta})^{1/2}$. This exhibits a very special feature of the ellipticals; namely, that matrix transformation (along with the radial R) suffices to characterize dependence. Further properties unique to the ellipticals are that the 2-spectral measure need not be unique (for $\alpha < 2$, however, all the α-spectral measures of a spherical distribution must be uniform on the unit sphere); furthermore, only for the ellipticals we have that the probability measure exhibits the same elliptical symmetry (affinely transformed isotropy) as the characteristic function; and as mentioned, only for the elliptical class there are distributions integrable at the order of the embedding index.

A special subclass of the pseudoisotropy, generalizing the sphericals, is the so-called α-symmetric distributions, introduced by Cambanis et al. [24], which exist for $\alpha \in (0, 2]$.

Definition 8. A pseudoisotropic $\mathbf{Z}$ is called *α-symmetric* or *standard α-symmetric* if one can take ς as the standard α-norm $\|\boldsymbol{\theta}\|_\alpha = (\sum_i |\theta_i|^\alpha)^{1/\alpha}$ (by slight abuse of notation). one then calls $\mathbf{X} = \boldsymbol{\Sigma}\mathbf{Z}$ *transformed α-symmetric*, or *$\boldsymbol{\Sigma}$-transformed α-symmetric*. one calls α (coinciding with the embedding index!) the *index of symmetry*.

Thus a transformed α-symmetric has α-spectral measure supported by only $2D$ unit vectors $\pm\mathbf{x}^{(1)}, \ldots, \pm\mathbf{x}^{(D)}$ which span $\mathbb{R}^D$; for the standard α-symmetrics, we have $\mathbf{x}^{(i)} = \mathbf{e}_i$.

Apart from the ellipticals, the arguably best known examples are the vectors of iid symmetric α-stables. Such a distribution has, if normalized to unit scale, characteristic function $\exp(-\|\boldsymbol{\theta}\|_\alpha^\alpha)$. More generally, it is known since Paul Lévy that there exist α-symmetric $\overline{\alpha}$-stable distributions iff $2 \geq \alpha \geq \overline{\alpha} > 0$ (with characteristic function form $\exp(-\|\boldsymbol{\theta}\|_\alpha^{\overline{\alpha}})$; such one can be generated by scaling an α-symmetric α-stable $\mathbf{X}$ by an independent radial R such that R^α is $\overline{\alpha}$-stable). The reader should beware the confusion in the literature, where the notion of symmetry most often in the modern literature means antipodal symmetry (as in this paper), although it is used by other authors in the past for rotational invariance (isotropy, implying ellipticity). This translates to a confusion as to whether the canonical choice for a multidimensional version of a symmetric $\overline{\alpha}$-stable is the one with iid coordinates ($\overline{\alpha}$-symmetric), or the elliptical one (chf $= \exp(-(\boldsymbol{\theta}^\top\boldsymbol{\theta})^{\overline{\alpha}/2})$), for example, in Owen and Rabinovitch [7, footnote 4]. Generally, when it comes to stable laws, the reader should be warned against the literature's inconsistent language and notation, dubbed by Hall [34] as a "comedy of errors."

For some α-symmetric distributions *not* generated from stables, see Gneiting [35].

5. Portfolio Separation with Risk-Free Investment Opportunity

The symmetry and positive homogeneity of the ς functional immediately yield two-fund monetary separation for the pseudoisotropics, in much the same way as the elliptical case or the case of linearly transformed iid α-stable components treated already by Fama [16]. It is already known that the independence of (linearly transformed) coordinates is not essential, for example, [36–38]. For $\alpha \leq 1$, the L^α unit ball is not only a nonconvex set; indeed, its complement intersected with any orthant is a convex set (the first-orthant part of the epigraph defining any component as a convex function of the others). This motivates the nondiversification final part of the following result.

Theorem 9. *Consider market (1) with the restriction that the portfolios are restricted to some closed radial set H (possibly = the entire $\mathbb{R}^D$). Suppose that conditionally on X_0, $\mathbf{X}$ is pseudoisotropic with standard ς.*

Then there is two-fund monetary separation: for any given $\boldsymbol{\xi}$, the return is first-order stochastically dominated by the return using portfolio $\boldsymbol{\xi}^* = \varsigma(\boldsymbol{\xi})\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}$ solves

$$\begin{aligned} &\max_{\boldsymbol{\xi}\in H} \quad \boldsymbol{\xi}^\top \boldsymbol{\mu} \\ &\text{subject to} \quad \varsigma(\boldsymbol{\xi}) = 1. \end{aligned} \tag{5}$$

Suppose in the following that H is a convex set such that $\boldsymbol{\xi}^\top\boldsymbol{\mu} \neq 0$ for some $\boldsymbol{\xi} \in H$. Then $\boldsymbol{\varphi}$ is unique if the ς-unit ball is a strictly convex set, in particular if the embedding index is > 1. On the other hand, if there is an extremum on an axis then it is optimal to only invest in one opportunity, either a positive position in the one with highest excess return/dispersion ratio, or shorting the one with the largest negative such; this in particular occurs with α-symmetric distributions when $\alpha \leq 1$. Under the additional assumption of iid coordinates and $H = \mathbb{R}^D$, holding only one risky opportunity implies either α-symmetric α-stability or that all but one μ_i vanish.

Proof. K has continuous boundary, so $\boldsymbol{\varphi}$ exists by the extreme value theorem. Consider $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^* = \varsigma(\boldsymbol{\xi})\boldsymbol{\varphi}$. We have $\boldsymbol{\xi}^{*\top}\mathbf{X} \sim \boldsymbol{\xi}^\top\mathbf{X}$ and thus (by independence and nonnegativity) $\boldsymbol{\xi}^{*\top}(\boldsymbol{\mu}R_0 + \mathbf{X}R) \sim \boldsymbol{\xi}^\top(\boldsymbol{\mu}R_0 + \mathbf{X}R) + (\boldsymbol{\xi}^* - \boldsymbol{\xi})\boldsymbol{\mu}R_0$, identifying the latter a.s. nonnegative (since $R_0 \geq 0$) term as $\widehat{X}$ in (2). Suppose H is convex. If the embedding index is > 1 the ς-unit sphere is smooth, yielding unique maximum unless (contrary to assumption) H is orthogonal to $\boldsymbol{\mu}$. Otherwise, we can have corner solutions; in particular for the standard α-quasinorms with $\alpha \leq 1$, the convex hull of the unit ball is the standard 1-norm, and pushing a plane as far as possible in one direction while intersecting this leads to a corner. Finally, as independent coordinates of a pseudoisotropic imply stability, iid coordinates imply α-symmetric α-stability, which is a well-known case. □

A comment is appropriate. From a first course in finance, one observes that agents will diversify and that if we introduce a new investment opportunity which offers return exceeding the risk-free, one will buy a positive amount of it as long as the hedging benefit of shorting (from a positive correlation) is not too large. Of course, the argument is based on some degree of integrability, and it is long known that nonintegrability *may* lead to plunging all eggs into one basket (and from the literature's focus on the iid coordinate case, e.g., Fama [16], this behaviour often shows up). It is generally not straightforward to describe the dependence structure outside L^2 (nor excess return outside L^1) but in the pseudoisotropic case, the location $\boldsymbol{\mu}$ and the ς-ball will reveal what we need to know. A sketch in $\mathbb{R}^2$ with $\mu_2 > \mu_1 > 0$ with the "unit sphere" being an ellipse around the origin will interpret analogues of "correlation" and "hedging" graphically, but the geometric arguments work for nonelliptical smooth ς-unit spheres too; if the ς-sphere through $(0, 1)$ does not fall too steeply there, we must have adaptation in the first quadrant. However, if we have a corner at $(0, 1)$, the situation is different, in particular, if ς is the 1-norm $|x_1|+|x_2|$. But even for nonintegrable cases we have differentiability if they admit embedding in L^p for some $p > 1$. Section 7 will utilize these arguments for the CAPM.

6. Some Special Results under Constrained Leverage or No Risk-Free Investment Opportunity

This section assumes transformed α-symmetry. By Assumption 1, we can take the linear transformation $\boldsymbol{\Sigma}$ of Definition 8 to be invertible even under constraint of type (7) below; should the constraint remove an unrestricted arbitrage, with $\boldsymbol{\Sigma}^\top\boldsymbol{\theta} = \mathbf{0}$ for some $\mathbf{0} \neq \boldsymbol{\theta} \perp \boldsymbol{\eta}$, this risk-free investment opportunity would violate Assumption 1.

Assumption 10. Throughout this section, $\mathbf{X}$ is $\boldsymbol{\Sigma}$-transformed α-symmetric: $\mathbf{X} = \boldsymbol{\Sigma}\mathbf{Z}$ for some α-symmetric $\mathbf{Z}$ and some invertible $\boldsymbol{\Sigma}$. Introduce the notation

$$\begin{aligned} \boldsymbol{\zeta}^\top &= \boldsymbol{\xi}^\top\boldsymbol{\Sigma}, \\ \boldsymbol{\nu} &= \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \\ \boldsymbol{\eta} &= \boldsymbol{\Sigma}^{-1}\mathbf{1} \end{aligned} \tag{6}$$

so that $\boldsymbol{\xi}^\top(\boldsymbol{\mu} + \mathbf{X}) = \boldsymbol{\zeta}^\top(\boldsymbol{\nu} + \mathbf{Z})$ and the total invested risky amount becomes $\boldsymbol{\zeta}^\top\boldsymbol{\eta}$. The only portfolio constraints considered in this section are the following *leverage constraint* type, where $L = \{w\}$ means no risk-free opportunity:

$$\boldsymbol{\zeta}^\top\boldsymbol{\eta} \in L \quad (\text{possibly agent-dependent}). \tag{7}$$

In particular we have *no* further restriction to arbitrary radial H.

The ellipticals admit 2-fund separation without risk-free opportunity, and we will see that this generalizes, at the cost of additional funds, to the special α-values $\alpha = 1 + 1/\text{odd}$.

Theorem 11. *Consider the market under Assumption 10. Put*

$$d = \frac{1}{2} \cdot \frac{\alpha}{\alpha - 1} \tag{8}$$

and assume in parts (a)–(d) that $\alpha \in (1, 2]$ ($\Leftrightarrow$ $d \geq 1$), while in part (e) assume $\alpha \in (0, 1]$.

(a) *For $\alpha \in (1, 2]$, the minimum-dispersion portfolio for no risk-free opportunity, that is, the one which minimizes the dispersion ς subject to the constraint, is $\boldsymbol{\xi} = (\boldsymbol{\Sigma}^\top)^{-1}\boldsymbol{\zeta}$ with*

$$\boldsymbol{\zeta} = \frac{w}{\|\boldsymbol{\eta}\|_{2d}^{2d}}\boldsymbol{\eta}^{\langle 2d-1\rangle}. \tag{9}$$

(b) *Suppose $d \in \mathbb{N}$ (i.e., $\alpha = 1 + 1/odd$). Then we have $2d + 1$-fund monetary separation under constrained leverage and $2d$-fund separation if there is no risk-free investment opportunity. In both cases, the $2d$ risky funds are (with the convention $0^0 = 1$)*

$$\boldsymbol{\varphi}_j = \left(\boldsymbol{\Sigma}^\top\right)^{-1}\left(\eta_1^{j-1}\nu_1^{2d-j},\ldots,\eta_n^{j-1}\nu_n^{2d-j}\right)^\top. \tag{10}$$

Call a portfolio "efficient" if it is a linear combination of these $\boldsymbol{\varphi}_j$ and satisfies leverage constraint (7). When $d > 1$ (i.e., $\alpha \in \{4/3, 6/5, 8/7, \ldots\}$), the set of efficient portfolios is not convex except in degenerate cases (this in contrast with the case $\alpha = 2 = d + 1$).

(c) *Let the assumptions of part (b) hold, and consider the number*

$$\tilde{k} := \min\left\{2d, \text{ the number of distinct real } \frac{\eta_i}{\nu_i} \text{ values plus 1 if } \exists \bar{i};\ \eta_{\bar{i}} \neq 0 = \nu_{\bar{i}}\right\}. \tag{11}$$

If there is no risk-free opportunity, then there are $\tilde{k}$ linearly independent funds such that all risk-averse agents can choose optimally among them, while any proper subset of these funds will fail to satisfy some risk-averse agent.

If $\tilde{k} > 1$, still assuming no risk-free opportunity exists, these funds also suffice for all agents, not necessarily risk-averse. If $\tilde{k} = 1$, then all risk-averse agents will choose minimum-dispersion portfolio (9) (recovering the classical degeneracy of one-fund separation over risk-averse agents), while other agents require an arbitrary (nonnull) free portfolio in addition.

Under constrained leverage, we have $\min\{3, \tilde{k}+1\}$ *fund monetary separation; the above funds together with the risk-free are sufficient to satisfy any agent.*

(d) *Part (b) does not generalize to the case where $d - 1/2 \in \mathbb{N}$ (i.e., $\alpha = 1 + 1/even$); if we formally consider the funds of (10) with an odd number for $2d$, there are cases where some agent cannot be satisfied by these funds.*

(e) *Suppose in this part that $\alpha \in (0, 1]$. Then any agent holds the zero position in all but at most two opportunities (where in contrast to separation results, different agents may require different pairs). The minimum-dispersion portfolio for the equality constraint $\boldsymbol{\zeta}^\top\boldsymbol{\eta} = w$ can be chosen on one axis (possibly nonunique). This portfolio is chosen by all risk-averse agents in the special case where $\boldsymbol{\nu}$ is proportional to $\boldsymbol{\eta}$.*

Before proceeding to the proof, notice that the case, where (11) yields 1, is the only where an opportunity with $\eta_i = \nu_i = 0$ is not redundant. Indeed, if $\boldsymbol{\Sigma}$ is the identity, then (11) counts the number of different marginal distributions of nonzero excess returns; then if there are at least two, one with zero excess return (possibly desired by a non-risk-averse agent) can be generated as a linar combination.

Proof. In order not to be first-order dominated, any agent who chooses the level $\widehat{w} \in L$ for $\boldsymbol{\zeta}^\top\boldsymbol{\eta}$ (where $\widehat{w} = w$ is mandatory if there is no risk-free opportunity) and the level $\bar{\varsigma}$ for dispersion must choose a solution of the problem

$$\begin{aligned} &\max_{\boldsymbol{\zeta}} \quad \boldsymbol{\zeta}^\top\boldsymbol{\nu} \\ &\text{subject to} \quad \|\boldsymbol{\zeta}\|_\alpha^\alpha = \bar{\varsigma}^\alpha, \\ &\qquad\qquad \boldsymbol{\zeta}^\top\boldsymbol{\eta} = \widehat{w} \end{aligned} \tag{12}$$

with associated Lagrange condition to be used in what to follow:

$$\boldsymbol{\nu} - \lambda\boldsymbol{\eta} = \delta\alpha\boldsymbol{\zeta}^{\langle\alpha-1\rangle} \tag{13}$$

(mnemoniac: λ for the leverage constraint, δ for the dispersion constraint).

(a) For the minimum-dispersion portfolio, consider the problem

$$\begin{aligned} &\min_{\boldsymbol{\zeta}} \quad \|\boldsymbol{\zeta}\|_\alpha \\ &\text{subject to} \quad \boldsymbol{\zeta}^\top\boldsymbol{\eta} = w \end{aligned} \tag{14}$$

and in case $\alpha > 1$ this is a concave problem with solution uniquely given by (9) (which is a limiting case of (13)).

(b) To cover the last part first, (13) has only two parameters, so the possibly optimal portfolios will, at least piecewise, form at most 2-dimensional surface, not convex unless subset of a plane, which, for the $2d$-fund cases, $d > 1$, requires the number of funds to degenerate to at most 3. The possible degeneracies are addressed in item (c).

Now for the separation result itself, odd signed powers are just ordinary powers, so (13) yields

$$\zeta_i \cdot (\delta\alpha)^{2d-1} = \left(\nu_i - \lambda\eta_i\right)^{2d-1}. \tag{15}$$

If $\delta \neq 0$, expand the power and collect terms to get the $2d$ risky funds given by (10), and in addition there is the risk-free, unless it vanishes identically. To address degeneracies, the constraint qualification could fail, but only at the minimum-dispersion portfolio, which is the fund $\boldsymbol{\varphi}_{2d}$. And the remaining case $\delta = 0$ implies $\boldsymbol{\nu} = \lambda\boldsymbol{\eta}$, which is subsumed in the next item.

(c) Let us first cover the case when $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ are proportional. Then the left-hand side of (13) collapses to one vector, a scaling of (9). In addition there is the risk-free, if one such exists, but if it does not, then by proportionality the excess return is uniquely given by w, so that $\delta = 0$.

If $\delta = 0$, there has to be an additional fund $\perp \boldsymbol{\eta}$, at zero price, but also not contributing to excess return, to satisfy agents who want higher than minimum dispersion. (Risk-averse agents will choose the zero position in this fund.)

To establish the number of funds needed, that is, the number of linearly independent vectors in expansion (10) (cf., (15)), assume $\delta \neq 0$ (otherwise we have the proportionality just covered) and λ such that

$$(\delta\alpha)^{2d-1}\boldsymbol{\zeta} = \sum_{j=1}^{2d}\binom{2d-1}{j-1}(-\lambda)^{j-1}\boldsymbol{\varphi}_j. \tag{16}$$

We wish to pick $2d$ agents with distinct λ values. That is possible (cf., Remark 3) as two distributions with different dispersions are never ordered by first-order dominance; dot each side of (13) with $\boldsymbol{\eta}$ to eliminate δ and get, for $w > 0$,

$$\boldsymbol{\zeta} = \frac{w}{\boldsymbol{\eta}^{\top}(\boldsymbol{\nu}-\lambda\boldsymbol{\eta})^{\langle 2d-1\rangle}}(\boldsymbol{\nu}-\lambda\boldsymbol{\eta})^{\langle 2d-1\rangle}. \tag{17}$$

Scaling the problem by w by replacing $\bar{\varsigma}$ by $w\widehat{\varsigma}$, we have a static maximization problem where different choices of dispersion lead to different λ's.

Gather the $2d$ agents' portfolios in a matrix $\boldsymbol{\Xi}$. Then we can write

$$\alpha^k\boldsymbol{\Xi}\boldsymbol{\Delta} = \boldsymbol{\Gamma}\boldsymbol{\Pi}\boldsymbol{\Lambda}^{\top}, \tag{18}$$

where $\boldsymbol{\Delta}$ is the diagonal $2d \times 2d$ invertible matrix with the agents' δ multipliers on the main diagonal, $\boldsymbol{\Pi}$ is the diagonal $2d \times 2d$ invertible matrix with the binomial coefficients $\binom{2d-1}{j-1}$ on the main diagonal, $\boldsymbol{\Gamma}$ is the matrix of the funds $(\boldsymbol{\varphi}_1,\ldots,\boldsymbol{\varphi}_{2d})$, and $\boldsymbol{\Lambda}$ is the Vandermonde matrix of the $(-\lambda)$'s, that is, with row j being the geometric sequence $(1,-\lambda_j,(-\lambda_j)^2,\ldots,(-\lambda_j)^{2d-1})^{\top}$, and invertible as the λ's are distinct.

It remains to find the rank of $\boldsymbol{\Gamma}$, and it follows by properties of Vandermonde determinants and their minors. Pick $\ell \leq 2d$ rows each with ν_i nonzero; these rows are then ν_i^{2d-1} times a geometric sequence $(1,\eta_i/\nu_i,\ldots,(\eta_i/\nu_i)^{2d-1})$, and we have full rank whenever these rows have η_i/ν_i distinct but not if two such ratios coincide. Let ℓ be the maximum number of linearly independent $\nu_i \neq 0$ rows, and form a matrix of these rows and an arbitrary nonnull row of the form $(0,\ldots,0,\eta_{\bar{\imath}}^{2d})$ (equivalent to $\nu_{\bar{\imath}} = 0 \neq \eta_{\bar{\imath}}$), if there is one. If such a nonnull row does exist and $\ell < 2d$ (= the number of columns), it is another linearly independent row.

The last statement follows as the unconstrained optimum is spanned by (10), namely, the single fund $\boldsymbol{\varphi}_1$.

(d) This part will implicitly use Remark 3 so that a continuum of multiplier pairs will actually be chosen by different agents. Observe that, in the even-power case, (13) does not yield (15), but

$$\zeta_i = \left(\frac{\nu_i-\lambda\eta_i}{\delta\alpha}\right)^k \operatorname{sign}\left(\frac{\nu_i-\lambda\eta_i}{\delta\alpha}\right) \tag{19}$$

which does not expand to a polynomial. Suppose for a counterexample that $\nu_n/\eta_n > \cdots > \nu_1/\eta_1 > 0$, with all $\eta_i > 0$. Let $\bar{\varsigma}$ grow from minimum dispersion (which is of the form of expansion (10)). At the point where the optimum falls outside the appropriate simplex (e.g., the unit simplex if $\boldsymbol{\eta} = \mathbf{1}$ and $w = 1$), opportunity #1 is shorted, requiring one more fund.

(e) Finally, assume $\alpha \in (0,1]$. Then the intersection of each orthant with the *exterior* of the L^{α} unit sphere is convex. Except in the proportional case, and as long as dimension exceeds 2, maximizing $\boldsymbol{\zeta}^{\top}\boldsymbol{\nu}$ subject to being in the plane $\boldsymbol{\zeta}^{\top}\boldsymbol{\eta}$ and on the L^{α} (quasi-norm) sphere is to move a line in parallel to this plane until it no longer intersects the interior of the L^{α} ball; then some coordinate becomes zero. Remove that coordinate from the model and repeat the argument until there are only two left (in which case the constraints form a discrete set and the process cannot be iterated).

Notice that the only way an agent can obtain dispersion as low as $c(\boldsymbol{\zeta}) = |w|/\max_i|\eta_i|$ is to choose all coordinates of $\boldsymbol{\zeta}$ as zero except for a (not necessarily unique) i with highest $|\eta_i|$ (nonzero, as the $\boldsymbol{\Sigma}$ matrix is assumed invertible), in which the position should be w/η_i; note that in case of nonuniqueness, the minimum dispersion is not attained by mixing two opportunities, except in the case $\alpha = 1$. This resolves the special case. Obviously, a minimum-dispersion portfolio is indispensable, as some agent would choose minimum dispersion. However, an agent choosing higher dispersion could very well choose two different opportunities, as the minimum-dispersion portfolio may not pay off very well in terms of ν_i (say, it could be zero).

□

Remark 12. In item (c) the first part asserts that all funds are needed in order to satisfy all agents, even all risk-averse, but the last sufficiency claim does not; although any level of dispersion will be chosen by some agent, it is not necessary so that any $\widehat{w}$ will be chosen. Assume, with no claim to realism, that all $\nu_i < 0$; then the opportunities will be shorted, and any (positive) upper bound on $\boldsymbol{\zeta}^{\top}\boldsymbol{\eta}$ would be inactive.

Using the leverage constraint, we can extend the separation result to agent-specific leverage-dependent interest rates as follows. Suppose that agent number a has interest spread of $r_a = r_a(\boldsymbol{\xi}^{\top}\mathbf{1}) = r_a(\boldsymbol{\zeta}^{\top}\boldsymbol{\eta})$ relative to the risk-free opportunity; intuitively it makes sense that r_a has the same sign as $\boldsymbol{\zeta}^{\top}\boldsymbol{\eta} - w$ (if it is interest *paid*). Then the agent's excess return at leverage $\widehat{w}$ is not anymore $\boldsymbol{\zeta}^{\top}\boldsymbol{\nu}$, but

$$\boldsymbol{\zeta}^{\top}\boldsymbol{\nu} - r_a\boldsymbol{\zeta}^{\top}\boldsymbol{\eta}. \tag{20}$$

The following property then easily carries over from the classical case.

Corollary 13. *Theorem 11 applies to the case of individual leverage-dependent interest rate just as for constrained leverage. Also, it admits $L = L_a$ individual.*

Proof. For whatever choice of $\overline{\varsigma}$, $\widehat{w}$ agent a considers, the $-r_a\boldsymbol{\zeta}^\top\boldsymbol{\eta} = -r_a(\widehat{w})\widehat{w}$ term goes outside the maximization, and the problem reduces to the problem for an agent with wealth $w = \widehat{w}$, choice $\overline{\varsigma}$, and no risk-free opportunity, except that agent a's position in the risk-free opportunity does not vanish. □

7. When Do We Have a Capital Asset Pricing Model?

This section establishes a Capital Asset Pricing Model for the pseudoisotropic distributions provided the embedding index exceeds 1, and there is a risk-free opportunity. We will in this case obtain some elements of the elliptical CAPM: there is the Markowitz bullet (namely, a (strictly!) convex risk/return set for the risky opportunities), a pricing characterization formula in the form of risk free return plus *beta* times market excess return times, and a securities market line where the agents will adapt.

A CAPM can be deduced *assuming* tradeoff between excess return ($\boldsymbol{\xi}^\top\boldsymbol{\mu}$, desired) and some dispersion functional, and so there is nothing novel to the following derivation save for the fact that (shifted) pseudoisotropy makes the location-dispersion ansatz valid for all agents. Apart from that, the argument mimics a textbook approach and we only sketch it: starting from a position in a location-dispersion efficient portfolio $\boldsymbol{\xi}^* \neq \mathbf{0}$ (same for all agents, up to scaling) the agent can then consider buying a (sufficiently small) portfolio $\boldsymbol{\delta}$ and scale the risky portfolio $\boldsymbol{\xi}^*$ by a factor $1 - b(\boldsymbol{\delta})/\varsigma(\boldsymbol{\xi}^*)$ as to maintain the portfolio returns dispersion $\varsigma(\boldsymbol{\delta} + (1 - b(\boldsymbol{\delta})/\varsigma(\boldsymbol{\xi}^*))\boldsymbol{\xi}^*)$ fixed at level $\varsigma(\boldsymbol{\xi}^*)$, this implicitly defining b. Formal differentiation yields $\nabla b(\mathbf{0}) = \nabla\varsigma(\boldsymbol{\xi}^*) =: \boldsymbol{\beta}^\top$. By the assumed efficiency, $\boldsymbol{\delta} = \mathbf{0}$ must maximize location given dispersion, yielding the formal first-order condition $\boldsymbol{\mu}^\top = (\boldsymbol{\mu}^\top\boldsymbol{\xi}^*/\varsigma(\boldsymbol{\xi}^*))\nabla b(\mathbf{0})$. Without rigorously defining "CAPM," we give the following stylized fact.

Proposition 14 (location-dispersion CAPM). *Suppose that the excess returns are $\boldsymbol{\mu} + \mathbf{X}$ and a risk-free opportunity exists and that each agent chooses portfolio (unrestricted) as to trade off the value of $\boldsymbol{\xi}^\top\boldsymbol{\mu}$ (of which more is preferred) against only a dispersion measure $\varsigma(\boldsymbol{\xi})$ which is positive for $\boldsymbol{\xi} \neq \mathbf{0}$ and homogeneous of degree one and has a subdifferential S at the market portfolio $\boldsymbol{\xi}^*$ (defined as the total risky investment made in the economy, or, by homogeneity, an arbitrary positive scaling).*

Then we have a CAPM with excess returns satisfying $\boldsymbol{\mu} = (\boldsymbol{\xi}^{*\top}\boldsymbol{\mu}/\varsigma(\boldsymbol{\xi}^*))\boldsymbol{\beta}$, for some $\boldsymbol{\beta} \in S$.

The hypothesis of this assertion is however a theorem under the assumption of pseudoisotropy and embedding index above 1; then ς is C^1 outside $\mathbf{0}$, and each agent (by the definition of "agent" in this paper) will choose a nondominated portfolio return, which by Theorem 9 is unique up to scaling. We summarize the following.

Theorem 15 (pseudoisotropic CAPM when the embedding index exceeds 1). *Consider a market of agents trading a given supply of risky opportunities with excess returns $\boldsymbol{\mu} + \mathbf{X}$ and one risk-free opportunity, where $\mathbf{X}$ is pseudoisotropic with standard $\varsigma \in \mathsf{C}^1(\mathbb{R}^D \setminus \{\mathbf{0}\})$ (in particular: if the embedding index is > 1).*

Then the hypothesis of Proposition 14 applies, with S being a singleton. If $\mathbf{X}$ is furthermore $\boldsymbol{\Sigma}$-transformed α-symmetric for $\alpha > 1$, then the betas are (uniquely) given as

$$\boldsymbol{\beta} = \boldsymbol{\Sigma}\left(\frac{\boldsymbol{\Sigma}^\top\boldsymbol{\xi}^*}{\left\|\boldsymbol{\Sigma}^\top\boldsymbol{\xi}^*\right\|_\alpha}\right)^{\langle\alpha-1\rangle}. \tag{21}$$

Observe that we cannot obtain the so-called zero-beta CAPM where no risk-free opportunity exists, as we do not have two-fund separation; we cannot then claim that a market aggregate of (agents' individual) efficient portfolios is efficient.

Remark 16. CAPM versions valid for integrable symmetric stable $\mathbf{X}$ are recovered as corollaries: first, the symmetric-stable CAPM of Fama [23], who assumed $\boldsymbol{\Sigma}$-transformed iid's, is precisely Theorem 15 with the additional assumption that $\alpha > 1$ is the index of stability *and* of symmetry.

Also we obtain and generalize even within the class of symmetric-stables the CAPM of Belkacem et al. [37] and of Gamrowski and Rachev [38]. Their approach employs the *covariation* (see [26, Section 2.7]) which unlike covariance is not symmetric: we speak of the covariation of a security's return *on* (not "and"!) the market portfolio's return; dividing this quantity by the dispersion as quantified by the standard $\varsigma(\cdot)$, we get a nonsymmetric "correlation coefficient" which becomes the security's *beta*. Indeed, formulating Theorem 15 in terms of the embedding index and the shape of ς extends not only the CAPM of [37, 38] but also the covariation function itself both beyond symmetric stability and to many non-integrable stable cases. The latter is not surprising though: the result for an α-symmetric α-stable $\mathbf{X}$ and $R^{\overline{\alpha}}$ being $\overline{\alpha}$-stable ($\overline{\alpha} \leq 1 \leq \alpha$) should not change if we instead consider $\mathbf{X}'R'$ where $R' \equiv 1$ and $\mathbf{X}' := \mathbf{X}R$, which is α-symmetric $\overline{\alpha}$-stable.

8. Outline: Dynamic Models Inheriting the Separation Properties of the Static Model

The results generalize to dynamic models where the price processes have the appropriately distributed increments. For a motivating example, reconsider the single-period market treated this far as a two-stage decision with preferences over initial consumption, terminal consumption, and bequest (= whatever remains), with an initial investment decision. Should a dominance result like Theorem 9 apply, the agent can improve a strategy that uses portfolio $\boldsymbol{\xi}$ the same way: keep initial consumption, replace $\boldsymbol{\xi}$ by $\boldsymbol{\xi}^* = \varsigma(\boldsymbol{\xi})\boldsymbol{\varphi}$, and, rather than keeping the excess wealth, increase terminal consumption by $(\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top\boldsymbol{\mu}R_0$, leaving the bequest unchanged in distribution. This should be preferable to an agent who prefers more to less, and we will adapt the preference assumption to the dynamic setting by modifying the mass-transfer criterion, for brevity omitting risk aversion in this section. The approach is based on [21], which in turn is based on an approach of Khanna and Kulldorff [39] which makes a somewhat less rigorous (but very neat!) argument for a geometric Brownian Black-Scholes-type model.

We take as given a sequence $t_0 < t_1 < t_2 < \cdots$, and a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ equipped with a filtration generated by a sequence of independent $\mathbf{Z}(t_n)$; $\mathscr{F}_0$ is generated by the null sets of $\mathscr{F}_0$ and inductively $\mathscr{F}_n$ by $\mathscr{F}_{n-1}$ and $\mathbf{Z}(t_n)$. Fixing initial *cumulative* consumption to zero without loss of generality and initial (but postconsumption!) wealth at given arbitrary y_0, we extend the mass-transfer concept of stochastic dominance over the probability laws of consumption-wealth process pairs (Y, γ) as follows.

Definition 17. By an *agent* we mean a pair $(y_0, \succeq)$ of wealth $y_0 \in \mathbb{R}$ at time t_0 and a partial ordering $\succeq$ over adapted process pairs such that (Y^*, γ^*) is weakly preferred over (Y, γ) whenever there is a.s. nondecreasing adapted process $\widehat{\gamma}$ with $\widehat{\gamma}(t_0) = 0$ such that $(Y^*, \gamma^*) \sim (Y, \gamma + \widehat{\gamma})$.

By a *strategy* we mean an adapted process pair $\{(\gamma(t_n), \boldsymbol{\xi}(t_n))\}_{n=0,1,\ldots}$ with $\gamma(t_0) = 0$, where we assume $\boldsymbol{\xi}(t_n)$ to take values in a set of precisely one of the two following forms (the choice of which is predetermined and nonrandom): either a given closed radial set H_n (if a risk-free opportunity exists) or to the set $\{\boldsymbol{\xi}(t_n)^\top \mathbf{1} = Y(t_n)\}$ for each agent (if it does not exist).

Remark that we do not want the agent to care about preconsumption wealth other than through consumption and postconsumption wealth, and Theorem 18 will be formulated accordingly, defining $Y(t_{n+1})$ net of the consumption at time t_{n+1}. Note also that as we only care about the law, that is, the finite-distributional distributions, we need only to show separation for every natural number of periods. It would be natural to restrict the strategies further, for example, requiring insolvent agents to close out their positions and stay on a fixed consumption per period (note that agents will become insolvent in this model); however, as we only compare strategies pairwise, we can show separation without any such admissibility restriction, and then discard any nonadmissible strategies, as long as the opportunity set only depends on past through the agent's history and is not restricted by increasing consumption from γ to γ^*.

As we have seen from the single-period model, the independent radial scalings in (1) play no part in the result, and neither does the distribution of the risk-free opportunity; we do not lose any generality by normalizing the radials to the constant 1 and the risk-free return to the constant 0.

We then have the dynamic model and the separation theorem as follows, which in the interest of brevity is formulated a bit loosely especially in part (b).

Theorem 18. *Suppose that each $\mathbf{Z}(t_{n+1})$ is pseudoisotropic with (nonrandom) standard ς_n. Assume given nonrandom $\boldsymbol{\mu}(t_n)$ and $\boldsymbol{\Sigma}(t_n)$, the latter satisfying for each agent and each strategy $\boldsymbol{\xi}(t_n)^\top \boldsymbol{\Sigma}(t_n) \mathbf{Z}(t_{n+1}) = \mathbf{0}$ only on the event $\{\boldsymbol{\xi}(t_n) = \mathbf{0}\}$ and possibly a null set. Suppose that discounted wealth at time t_m, $m \in \mathbb{N}$, is given by*

$$Y(t_m) = y_0 - \gamma(t_m) + \sum_{n=0}^{m-1} \boldsymbol{\xi}(t_n)^\top \left[\boldsymbol{\mu}(t_n) + \boldsymbol{\Sigma}(t_n) \mathbf{Z}(t_{n+1})\right]. \tag{22}$$

Assume that at time t_n the following hold with $\boldsymbol{\mu}(t_n)$ for $\boldsymbol{\mu}$ and $\mathbf{X} := \boldsymbol{\Sigma}(t_n)\mathbf{Z}(t_{n+1})$:

(a) *If there is a risk-free opportunity, assume the hypothesis of Theorem 9 is satisfied with $H :=$ the (radial) portfolio restriction at time t_n. Then for every agent in the dynamic market and each strategy $(\gamma, \boldsymbol{\xi})$, there is one strategy which at time t_n uses the portfolio $\varsigma(\boldsymbol{\xi}(t_n))\boldsymbol{\varphi}$, and which leads to a preferred wealth-consumption process. $\boldsymbol{\varphi}$ (given in Theorem 9) is common to all agents.*

(b) *If there is no risk-free opportunity assume the hypothesis of some part (a) to (e) of Theorem 11 holds at time t_n. Then analogously, the conclusion of the respective part of Theorem 11 holds at time t_n (with the same funds, as therein).*

In words, this means that the dynamic model inherits, time-by-time, the separation properties that the distribution would infer in a single-period model.

A proof can easily be constructed from the proof of the single-period model by following [21], which shows the elliptical or stable case in the more complicated *continuous time* model, based on Khanna and Kulldorff [39] for the Gaussian case. The essence is that we can simply consume the excess *at each time*, and the (strong) Markov property will leave us with the same opportunity set for all future. To see this, consider a single time t_n for which the hypothesis holds true. Whatever portfolio $\boldsymbol{\xi}$ the strategy yields, we can replace it by some $\boldsymbol{\xi}^*$ of the same scale $\varsigma(\boldsymbol{\xi}^*) = \varsigma(\boldsymbol{\xi})$ which uses the fund. Imagine for the moment that we simply dispose of the excess $(\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu}(t_n)$ (≥ 0); then we have merely replaced next period's wealth and consumption by one of the same (conditional) distribution, and thus the opportunity set, the set of *possible* laws of $\{(Y^*, \gamma^*)\}_{t > t_n}$ (thus of $\{(Y^*, \gamma^*)\}_{t \geq t_0}$), is the same; by assumption this replacement is (weakly) preferred by every agent. Now drop the fictitious disposal and increase the consumption at time t_{n+1} by $(\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu}(t_n)$. By assumption, this increase is (weakly) preferred by every agent.

9. Concluding Remarks

It was natural to develop portfolio theory for shifted symmetric stable returns, from the defining property of stability, as long as one did not realize that matrix multiplication was not sufficient to capture the dependence structure. Indeed, with preferences only over portfolio return, not of the returns of the individual opportunities, the defining property of pseudoisotropic distributions almost begs the question of portfolio separation, and this paper has extended the classical portfolio theory to those distributions. Doing so, we are able to recast the theory for symmetric stables as well, in a way that not only is much less restrictive but shows that the basic properties associated with symmetric stables are not the crucial ones; the essential properties are in the geometry of the standard ς.

Within the class of pseudoisotropic distributions, the possible diversification properties that follow from suitable integrability are not integrability properties; they are geometric embeddability properties. For $\overline{\alpha}$-stables, the assumption that $\overline{\alpha} > 1$ is sufficient (but not necessary) that the index of embedding is > 1, which again is sufficient that the ς-spheres are smooth, from which the qualitative properties of the classical cases are recovered. Thus for the symmetric stables, it is not the index of stability that is crucial.

The class of (transformed) α-symmetric distributions highlights this: the α-symmetric stable distributions behave in some sense, the same for given α, no matter which index $\overline{\alpha}$ of stability. Inspecting Fama's characterization [16, formulae (14)–(17)] we note that they hold true if we merely replace his assumption of α-stability by an assumption of α-symmetry. This is well understood for the elliptical case (Owen and Rabinovitch [7]), and we have established a direct extension, under the assumption that a risk-free opportunity exists.

For the case without risk-free opportunity, the elliptical distributions have unique properties, and it is directly connected to the shape of the ς-spheres. For the (transformed) α-symmetric distributions, the geometry enables us to make a sequence of symmetry indices that admit weaker separation results, namely, k-fund separation result for $1 + 1/(k - 1)$-norm-symmetric variables when k is even, generalizing the elliptical case $k = 2$. Thus within the class of α-symmetric distributions ($\alpha > 1$), separation now looks like an exceptional property, one of a sequence inside a continuum, although it is a generalization of a property of a family widely considered an adequate approximation of reality (at least implicitly, ellipticity is necessary for linearity of regression in dimension > 2, Hardin Jr. [40]).

Although portfolio separation is a theoretical result which has historically not concerned fit to real data, we make a remark on applicability. Various heavy-tailed models have been introduced to find a better fit to data, though one can question whether a low order of integrability is in line with the real world. However, the exact *asymptotical* tail index is not necessarily the scope of application for a financial model. Indeed, with the emergence of quantile measures (the infamous *value-at-risk*), financial risk is often measured in a way that totally disregards the order of integrability. Not only does this make the objections less valid, one does not extract from the model the properties that are most questionable, but also the nonsubadditivity of value-at-risk may in certain cases penalize diversification; behaviour according to this does in fact *require* nonintegrability, and even that is not sufficient: in the pseudoisotropic model this translates into the ς-sphere having corner (nondifferentiability points) and an embedding index of at most one.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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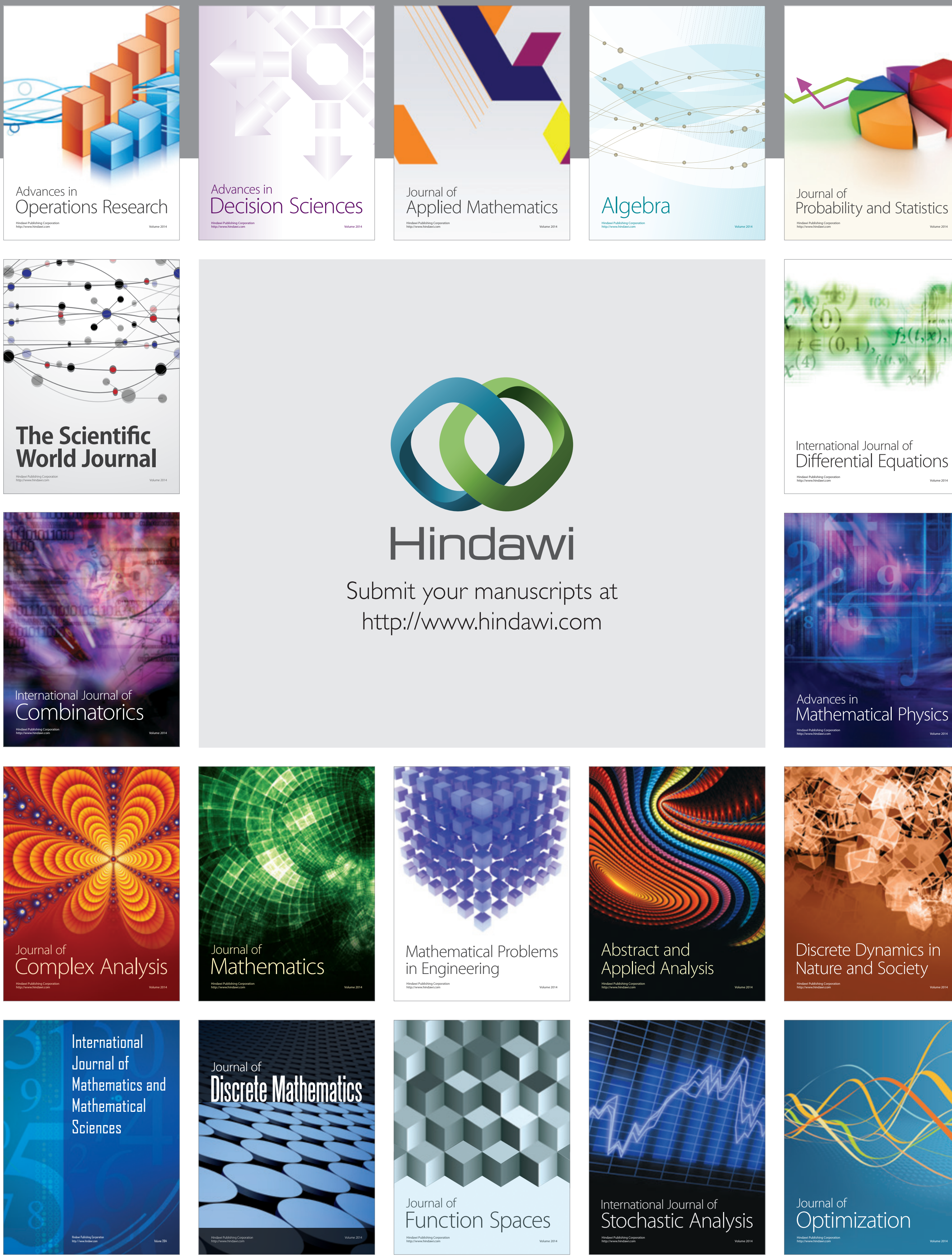
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